

MOTIVIC COHOMOLOGY OF THE COMPLEMENT OF HYPERPLANE ARRANGEMENTS

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ABSTRACT. We give a presentation of the motivic cohomology ring of the complement of a hyperplane arrangement considered as algebra over the motivic cohomology of the ground field.

INTRODUCTION

Let K be a field and $U \subset \mathbb{A}_K^N$ the complement of a finite union of hyperplanes. For $K = \mathbb{C}$ the cohomology ring $H^*(U^{an}, \mathbb{Z}_{U^{an}})$ of the constant sheaf $\mathbb{Z}_{U^{an}}$ is isomorphic to the subalgebra of meromorphic forms generated by the logarithmic forms $\frac{1}{2\pi i} \frac{df}{f}$ ([Ar],[Br]). Motivic cohomology $H(U) := \oplus_{p,q} H^p(U, \mathbb{Z}(q))$ as defined by Voevodsky [Vo] is defined over an arbitrary field and there is a natural isomorphism $\mathbb{G}_m(U) \cong H^1(U, \mathbb{Z}(1))$. The aim of this article is to give a description of the motivic cohomology ring $H(U)$ for a perfect ground field K . We describe the module structure over $H(K)$ and the ring structure.

Concerning the module structure, we show that $H(U)$ is a free $H(K)$ module (Corollary 1.7) and give a combinatorial description for the rank (Corollary 3.11). For the ring structure, we consider $H(U)$ as an algebra over $H(K)$. We prove that $H(U)$ is generated by the units $\mathbb{G}_m(U)$ and give the relations. We have to introduce some notation to describe the relations and to state the theorem. Let $H(K)\{\mathbb{G}_m(U)\}$ be the free, bigraded commutative algebra over $H(K)$ generated by the abelian group of units of U (in degree $(1,1)$). If $f \in \mathbb{G}_m(U)$ then we denote by (f) the corresponding element in $H(K)\{\mathbb{G}_m(U)\}$ and for $\lambda \in K^\times$ we denote by $[\lambda] \in H^1(K, \mathbb{Z}(1)) \subset H(K)$ the associated cohomology class.

Theorem (Theorem 3.5). *Let K be a perfect field. The morphism*

$$H(K)\{\mathbb{G}_m(U)\}/I \rightarrow H(U),$$

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defined by mapping (f) to the class $[f]$ of f in $H^1(U, \mathbb{Z}(1))$, is an isomorphism of $H(K)$ algebras, where the ideal I is generated by the elements:

$$\begin{aligned} (1) \quad & (f) - [f], \quad \text{if } f \in K^\times \subset \mathbb{G}_m(U), \\ (2) \quad & (f_1) \cdot (f_2) \cdots (f_t), \quad \text{if } f_i \in \mathbb{G}_m(U), i = 1, \dots, t, \text{ s.t. } \sum_{i=1}^t f_k = 1, \\ (3) \quad & (f)^2 + [-1] \cdot (f), \quad \text{if } f \in \mathbb{G}_m(U). \end{aligned}$$

Relation (1) is trivial, relation (2) and (3) are well-known in the Milnor K-theory of the function field $K(U)$ ([Mi], Lemma 1.2, 1.3).

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1. GENERATING CLASSES

1.1. Let K be a perfect field. We work with the triangulated category $DM_{gm}^{eff}(K)$ of effective geometrical motives over K , defined by Voevodsky ([Vo], Definition 2.1.1). We denote by M_{gm} the functor $Sm/K \rightarrow DM_{gm}^{eff}(K)$ that maps a smooth scheme to the corresponding motive. For a smooth scheme X of finite type over K motivic cohomology is defined as

$$H^p(X, \mathbb{Z}(q)) := \mathrm{Hom}_{DM_{gm}^{eff}}(M_{gm}(X), \mathbb{Z}(q)[p]).$$

Let $U \subset \mathbb{A}_K^N$ be the complement of a finite union of hyperplanes.

Proposition 1.1. *There is an isomorphism*

$$M_{qm}(U) \cong \bigoplus_{i \in I} \mathbb{Z}(n_i)[n_i]$$

for some finite index set I and integers $n_i \geq 0$.

Proof. Let Y_1, \dots, Y_r be the hyperplanes in the complement of U , i.e. $U = \mathbb{A}^N - \cup_{i=1}^r Y_i$. We proceed by induction on r . If $U = \mathbb{A}_K^N$ then $M_{gm}(U) = \mathbb{Z}$. If $r \geq 1$ then the assertion is true for $U' := \mathbb{A}^N - \cup_{i=2}^r Y_i$ and $Y_1 \cap U' = Y_1 - \cup_{i=2}^r (Y_1 \cap Y_i)$ by induction. Consider the Gysin triangle

$$(1.2) \quad M_{gm}(U) \rightarrow M_{gm}(U') \xrightarrow[2]{\phi} M_{gm}(Y_1 \cap U')(1)[2] \xrightarrow{+1}.$$

We have $\phi = 0$ since

$$(1.3) \quad \text{Hom}_{DM_{gm}^{eff}}(\mathbb{Z}(n)[n], \mathbb{Z}(m)[m+1]) = 0 \quad \text{for every } n, m \geq 0,$$

which is proved in Lemma (1.5) below. It follows that the sequence (1.2) is split and there is a non-canonical isomorphism

$$(1.4) \quad M_{gm}(U) \cong M_{gm}(U') \oplus M_{gm}(Y_1 \cap U')(1)[1],$$

which proves the assertion. \square

Lemma 1.5. *For every $n, m \geq 0$ the identity (1.3) holds.*

Proof. By the Cancellation Theorem ([Vo2], Corollary 4.10) we may reduce to

$$\begin{aligned} \text{Hom}_{DM_{gm}^{eff}}(\mathbb{Z}(n)[n], \mathbb{Z}[1]) &= 0, \quad n \geq 0 \\ \text{Hom}_{DM_{gm}^{eff}}(\mathbb{Z}, \mathbb{Z}(n)[n+1]) &= 0, \quad n \geq 0. \end{aligned}$$

The first group is a direct summand of

$$\text{Hom}_{DM_{gm}^{eff}}(M_{gm}(\mathbb{G}_m^{\times n}), \mathbb{Z}[1]) = H_{\text{Zar}}^1(\mathbb{G}_m^{\times n}, \mathbb{Z}) = 0;$$

the second group is a direct summand of $H_{\text{Zar}}^1(K, C_*(\mathbb{G}_m^{\times n}))$, which is trivial because $C_*(\mathbb{G}_m^{\times n})$ is concentrated in non-positive degrees. \square

The splitting of the Gysin sequence (1.2) yields the *split* short exact sequence

$$(1.6) \quad 0 \rightarrow H^p(U', \mathbb{Z}(q)) \rightarrow H^p(U, \mathbb{Z}(q)) \rightarrow H^{p-1}(Y_1 \cap U', \mathbb{Z}(q-1)) \rightarrow 0.$$

Using induction again and the isomorphism (1.4) we have:

Corollary 1.7. *The motivic cohomology $H(U)$ is a finitely generated free module over the motivic cohomology of the ground field $H(K)$.*

1.2. A unit $f \in \mathbb{G}_m(U)$ gives a cohomology class $[f] \in H^1(U, \mathbb{Z}(1))$ defined by the morphism

$$M_{gm}(U) \xrightarrow{f} M_{gm}(\mathbb{G}_m) \xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z}(1)[1] \xrightarrow{\text{proj}} \mathbb{Z}(1)[1].$$

The decomposition $M_{gm}(\mathbb{G}_m) \cong \mathbb{Z} \oplus \mathbb{Z}(1)[1]$ is constructed from the splitting $\mathbb{Z} \xrightarrow{\cong} M_{gm}(K) \xrightarrow{\text{inclusion at } 1} M_{gm}(\mathbb{G}_m)$. By ([Vo], Corollary 3.5.3) the map $f \mapsto [f]$ is an isomorphism of abelian groups.

Proposition 1.8. *The cohomology ring $H(U)$ is generated by the classes of the units in U as an algebra over $H(K)$.*

Proof. We use the short exact sequence (1.6)

$$0 \rightarrow \oplus_{p,q} H^p(U', \mathbb{Z}(q)) \xrightarrow{\alpha} \oplus_{p,q} H^p(U, \mathbb{Z}(q)) \xrightarrow{\beta} \oplus_{p,q} H^{p-1}(Y_1 \cap U', \mathbb{Z}(q-1)) \rightarrow 0$$

of $H(K)$ modules. The map α is the restriction to U , hence it is a ring homomorphism which maps $[f]$ to $[f|_U]$ for $f \in \mathbb{G}_m(U')$. For the map β , let $t \in \mathbb{A}^1(U')$ be a function with vanishing locus $Y_1 \cap U'$ and denote by $\iota : H(U') \rightarrow H(Y_1 \cap U')$ the restriction morphism. I claim that

$$(1.9) \quad \beta([t] \cdot \alpha(z)) = \iota(z)$$

holds for any z . In DM_{gm}^{eff} we need to show that the composition

$$\begin{aligned} M_{gm}(Y_1 \cap U')(1)[1] &\xrightarrow{\text{Gysin}} M_{gm}(U) \xrightarrow{(incl, t)} M_{gm}(U' \times \mathbb{G}_m) \\ &\xrightarrow{id \otimes \text{proj}} M_{gm}(U')(1)[1] \end{aligned}$$

is equal to $M_{gm}(\text{inclusion})(1)[1]$. This follows from the morphism of Gysin triangles:

$$\begin{array}{ccccccc} M_{gm}(Y_1 \cap U')(1)[1] & \xrightarrow{+1} & M_{gm}(U) & \longrightarrow & M_{gm}(U') & \longrightarrow & \\ \downarrow \text{incl}(1)[1] & & \downarrow (incl, t) & & \downarrow (id, t) & & \\ M_{gm}(U')(1)[1] & \xrightarrow{+1} & M_{gm}(U' \times \mathbb{G}_m) & \longrightarrow & M_{gm}(U' \times \mathbb{A}^1) & \longrightarrow & \end{array}$$

and the commutative diagram

$$\begin{array}{ccc} M_{gm}(U')(1)[1] & \xrightarrow{\text{Gysin}} & M_{gm}(U' \times \mathbb{G}_m) \\ & \searrow = & \downarrow id_{U'} \otimes \text{proj} \\ & & M_{gm}(U')(1)[1]. \end{array}$$

In particular, if $f_1, \dots, f_s \in \mathbb{G}_m(U')$ then

$$\beta([t] \cdot [f_1] \cdots [f_s]) = [f_1|_{Y_1 \cap U'}] \cdots [f_s|_{Y_1 \cap U'}].$$

Since $\mathbb{G}_m(U') \rightarrow \mathbb{G}_m(Y_1 \cap U')$ is surjective, we conclude by induction on the number of removed hyperplanes that the algebra generated by $\mathbb{G}_m(U)$ maps via β surjectively onto $H(Y_1 \cap U')$ and contains $H(U')$, thus equals $H(U)$. \square

2. RELATIONS

2.1. Let U be a smooth K -scheme. The purpose of this section is to show that the following elements in the motivic cohomology of U are trivial:

(2.1)

$$[f_1] \cdot [f_2] \cdots [f_t] = 0, \quad \text{if } f_i \in \mathbb{G}_m(U), i = 1 \dots t, \text{ such that } \sum_{i=1}^t f_i = 1,$$

and

$$(2.2) \quad [f]^2 + [-1] \cdot [f] = 0, \quad \text{if } f \in \mathbb{G}_m(U).$$

To prove the identities, we reduce the general case to the case $U = \text{Spec}(K)$; here we use the comparison theorem of motivic cohomology and Milnor K -theory by Suslin and Voevodsky (see [SuVo])

$$(2.3) \quad H^n(K, \mathbb{Z}(n)) \cong K_n^M(K), \quad \text{for any } n \geq 0.$$

It is sufficient to prove (2.1) for the scheme $\Delta^t - \cup_{i=1}^t \{x_i = 0\}$, where $\Delta^t = \{\sum_i x_i = 1\} \subset \mathbb{A}^t$, and $f_1 = x_1, \dots, f_t = x_t$, because the general case follows by pullback. Similarly, it is enough to show (2.2) for \mathbb{G}_m . It is somewhat easier to prove the identity

$$(2.4) \quad R(f_1, \dots, f_t) = 0, \quad \text{if } f_i \in \mathbb{G}_m(U), i = 1 \dots t, \text{ such that } \sum_{i=1}^t f_i = 0,$$

where $R(f_1, \dots, f_t) :=$

$$\begin{aligned} & \sum_{i=1}^t (-1)^i [f_1] \cdots \widehat{[f_i]} \cdots [f_t] + \sum_{i < j} [-1] \cdot [f_1] \cdots \widehat{[f_i]} \cdots \widehat{[f_j]} \cdots [f_t] + \\ & \sum_{i < j < k} [-1]^2 \cdot [f_1] \cdots \widehat{[f_i]} \cdots \widehat{[f_j]} \cdots \widehat{[f_k]} \cdots [f_t] + \dots \end{aligned}$$

Lemma 2.5. *The identities (2.1) together with (2.2) are equivalent to the identity (2.4).*

Proof. First, assume (2.1) and (2.2), let f_1, \dots, f_t be units and $\sum_i f_i = 0$. Then $\sum_{i=1}^{t-1} \frac{f_i}{-f_t} = 1$ and

$$(2.6) \quad ([f_1] - [f_t] + [-1]) \cdots ([f_{t-1}] - [f_t] + [-1]) = 0$$

by (2.1). Using skew-commutativity: $[f][g] + [g][f] = 0$, for all units f, g , we see with the help of $[f_t]^2 - [-1] \cdot [f_t] = 0$ (2.2) and $[-1] + [-1] = 0$ that (2.6) is equal to $(-1)^t R(f_1, \dots, f_t)$.

Now we assume (2.4). Since $R(f_1, \dots, f_{t-1}, -1) = (-1)^t [f_1] \cdots [f_{t-1}]$ we get (2.1). In order to show (2.2) we may reduce to $U = \mathbb{G}_m$. If $K \neq \mathbb{F}_2$ then there exist units $\lambda_1, \lambda_2, \lambda_3 \in K^\times$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 0$; and we have

$$R(\lambda_1 f, \lambda_2 f, \lambda_3 f) = [f]^2 + [-1] \cdot [f] + R(\lambda_1, \lambda_2, \lambda_3).$$

For the case $K = \mathbb{F}_2$, we use $M_{gm}(\mathbb{G}_m) = \mathbb{Z} \oplus \mathbb{Z}(1)[1]$ to see $H^2(\mathbb{G}_m, \mathbb{Z}(2)) = H^2(\mathbb{F}_2, \mathbb{Z}(2)) \oplus H^1(\mathbb{F}_2, \mathbb{Z}(1))$, which is zero by the isomorphism with Milnor K -theory (2.3). \square

2.2. Let $U \subset \mathbb{A}_K^N$ be the complement of a finite union of hyperplanes Y_1, \dots, Y_r . We define $U_j := \mathbb{A}_K^N - \cup_{i \neq j}^r Y_i$.

The next lemma will serve as a criterion for a cohomology class to be trivial.

Lemma 2.7. *Any morphism $\phi : M_{gm}(U) \rightarrow T$ in DM_{gm}^{eff} such that*

$$M_{gm}(Y_j \cap U_j)(1)[1] \xrightarrow{Gysin} M_{gm}(U) \xrightarrow{\phi} T$$

is trivial for every $j = 1, \dots, r$, factors through $M_{gm}(K)$, i.e. there is a morphism $\psi : M_{gm}(K) \rightarrow T$ and a commutative diagram

$$\begin{array}{ccc} M_{gm}(U) & \xrightarrow{\phi} & T \\ \downarrow & \nearrow \psi & \\ M_{gm}(K) & & \end{array}$$

Proof. We prove by induction on r . The case $U = \mathbb{A}^N$ is obvious.

By assumption and with the help of the Gysin triangle (1.2) the morphism ϕ factors as $\phi : M_{gm}(U) \xrightarrow{M_{gm}(incl)} M_{gm}(U_1) \xrightarrow{\phi_1} T$ for some ϕ_1 . For every $j > 1$ the diagram

$$\begin{array}{ccccc} M_{gm}(U_j \cap Y_j)(1)[1] & \xrightarrow{Gysin} & M_{gm}(U) & \searrow \phi & \\ \downarrow M_{gm}(incl)(1)[1] & & \downarrow & \nearrow \phi_1 & T \\ M_{gm}((U_j \cup U_1) \cap Y_j)(1)[1] & \xrightarrow{Gysin} & M_{gm}(U_1) & & \end{array}$$

is commutative. Since $M_{gm}((U_j \cup U_1) \cap Y_j)$ is isomorphic (by the morphism $M_{gm}(incl)$) to a direct summand of $M_{gm}(U_j \cap Y_j)$ (1.4) it follows from $\phi_1 \circ Gysin \circ M_{gm}(incl)(1)[1] = \phi \circ Gysin = 0$ that $\phi_1 \circ Gysin = 0$. Now, we can apply induction to ϕ_1 . \square

Proposition 2.8. *Let U be a smooth K -scheme of finite type and $f_1, \dots, f_t \in \mathbb{G}_m(U)$ units such that $\sum_{i=1}^t f_i = 0$. Then*

$$R(f_1, \dots, f_t) = 0.$$

Proof. We may assume $U := H - \cup_{i=1}^t \{x_i = 0\}$, $H \subset \mathbb{A}^t$ the hyperplane $\sum_{i=1}^t x_i = 0$, and $f_i = x_i$ for every i , where x_1, \dots, x_t are the coordinates of \mathbb{A}^t .

Denote by $\beta_j : H^{t-1}(U, \mathbb{Z}(t-1)) \rightarrow H^{t-2}(Y_j \cap U_j, \mathbb{Z}(t-2))$, for every $j = 1, \dots, t$, the morphism from the Gysin sequence; here $Y_j = \{x_j = 0\}$ and $U_j = H - \cup_{i \neq j} \{x_i = 0\}$. Formula (1.9) immediately implies

$$\beta_j(R(x_1, \dots, x_t)) = (-1)^j R(x_1|_{Y_j \cap U_j}, \dots, \widehat{x_j}, \dots, x_t|_{Y_j \cap U_j}),$$

the righthand side being zero by induction on t . Lemma (2.7) implies that $R(x_1, \dots, x_t)$ is the pullback of some class in $H^{t-1}(K, \mathbb{Z}(t-1))$. If

$K \neq \mathbb{F}_2$ then U has a K rational point. After pullback to this point we have to prove: $R(\lambda_1, \dots, \lambda_t) = 0$ in $H^{t-1}(K, \mathbb{Z}(t-1))$ for $\lambda_i \in K^\times$ with $\sum_i \lambda_i = 0$. We may use the isomorphism (2.3) to work with Milnor K -theory. In Milnor K -theory the formulas (2.1), (2.2) are well-known ([Mi], Lemma 1.2, 1.3) and they yield $R(\lambda_1, \dots, \lambda_t) = 0$ as in the proof of Lemma (2.5). \square

With the help of Lemma (2.5) we have the following corollary.

Corollary 2.9. *Let U be a smooth K -scheme.*

- (1) *If $f_1, \dots, f_t \in \mathbb{G}_m(U)$ are units such that $\sum_{i=1}^t f_i = 1$ then $[f_1] \cdots [f_t] = 0$.*
- (2) *For every $f \in \mathbb{G}_m(U)$ we have $[f]^2 + [-1] \cdot [f] = 0$.*

3. THE COHOMOLOGY RING

3.1. Recall that $U \subset \mathbb{A}_K^N$ is the complement of a finite union of hyperplanes Y_1, \dots, Y_r .

We let $H(K)\{\mathbb{G}_m(U)\}$ be the free, bigraded commutative algebra over $H(K)$ generated by the abelian group of units of U in degree $(1, 1)$. For $f \in \mathbb{G}_m(U)$ we denote by (f) the corresponding element in $H(K)\{\mathbb{G}_m(U)\}$.

We denote by $I_U \subset H(K)\{\mathbb{G}_m(U)\}$ the ideal generated by the following elements:

$$(3.1) \quad (f) - [f], \quad \text{if } f \in K^\times \subset \mathbb{G}_m(U),$$

$$(3.2) \quad (f_1) \cdot (f_2) \cdots (f_t), \quad \text{if } f_i \in \mathbb{G}_m(U), i = 1, \dots, t, \text{ such that } \sum_{i=1}^t f_i = 1,$$

$$(3.3) \quad (f)^2 + [-1] \cdot (f), \quad \text{if } f \in \mathbb{G}_m(U).$$

As above we define $\tilde{R}(f_1, \dots, f_t) \in H(K)\{\mathbb{G}_m(U)\}$ to be the element

$$\begin{aligned} \sum_{i=1}^t (-1)^i (f_1) \cdots \widehat{(f_i)} \cdots (f_t) + \sum_{i < j} [-1] \cdot (f_1) \cdots \widehat{(f_i)} \cdots \widehat{(f_j)} \cdots (f_t) + \\ \sum_{i < j < k} [-1]^2 \cdot (f_1) \cdots \widehat{(f_i)} \cdots \widehat{(f_j)} \cdots \widehat{(f_k)} \cdots (f_t) + \dots \end{aligned}$$

By the same arguments as in the proof of Lemma (2.5) we can replace (3.2) in the definition of I_U by

$$(3.4) \quad \tilde{R}(f_1, \dots, f_t), \quad \text{if } f_i \in \mathbb{G}_m(U), i = 1, \dots, t, \text{ s.t. } \sum_{i=1}^t f_i = 0.$$

For every hyperplane Y_i we choose a polynomial ϕ_i of degree 1, which defines Y_i . To prove the main theorem it will be useful to work with the ideal $I'_U \subset I_U$ generated by elements of the form (3.1), (3.3) and elements as in (3.4) with $f_j = \lambda_j \phi_{i_j}$ or $f_j = \lambda_j$ for every j and some $\lambda_j \in K^\times$ and some index i_j .

Proposition (1.8) and Corollary (2.9) imply that we have surjective morphisms

$$H(K)\{\mathbb{G}_m(U)\}/I'_U \rightarrow H(K)\{\mathbb{G}_m(U)\}/I_U \rightarrow H(U),$$

defined by mapping (f) to $[f]$.

Theorem 3.5. *Let K be a perfect field. The morphism of $H(K)$ algebras*

$$H(K)\{\mathbb{G}_m(U)\}/I_U \rightarrow H(U),$$

is an isomorphism.

Proof. Obviously it is sufficient to prove the assertion for I'_U . The proof is done by induction on the number of removed hyperplanes.

Let $U' := \mathbb{A}_K^N - \bigcup_{i \geq 2} Y_i$, then by induction we have

$$\begin{aligned} H(K)\{\mathbb{G}_m(U')\}/I'_{U'} &\xrightarrow{\cong} H(U'), \\ H(K)\{\mathbb{G}_m(Y_1 \cap U')\}/I'_{Y_1 \cap U'} &\xrightarrow{\cong} H(Y_1 \cap U'). \end{aligned}$$

The restriction map $\mathbb{G}_m(U') \rightarrow \mathbb{G}_m(Y_1 \cap U')$ induces

$$\iota : H(K)\{\mathbb{G}_m(U')\}/I'_{U'} \rightarrow H(K)\{\mathbb{G}_m(Y_1 \cap U')\}/I'_{Y_1 \cap U'},$$

which is the pullback morphism in motivic cohomology for the inclusion $Y_1 \cap U' \rightarrow U'$. Furthermore, the restriction $\mathbb{G}_m(U') \rightarrow \mathbb{G}_m(U)$ induces

$$\tilde{\alpha} : H(K)\{\mathbb{G}_m(U')\}/I'_{U'} \rightarrow H(K)\{\mathbb{G}_m(U)\}/I'_U,$$

and we define $\tilde{\beta}$ to be the composition

$$H(K)\{\mathbb{G}_m(U)\}/I'_U \rightarrow \oplus_{p,q} H^p(U, \mathbb{Z}(q)) \rightarrow \oplus_{p,q} H^{p-1}(Y_1 \cap U', \mathbb{Z}(q-1)),$$

where the second arrow comes from Gysin sequence. To see how $\tilde{\beta}$ maps, we may write every element $x \in H(K)\{\mathbb{G}_m(U)\}$ as

$$x = (\phi_1) \cdot \tilde{\alpha}(x_1) + \tilde{\alpha}(x_2).$$

This can be done by using relation (3.3). Then formula (1.9) yields $\tilde{\beta}(x) = \iota(x_1)$.

Obviously we have a commutative diagram

$$\begin{array}{ccccc}
H(K)\{\mathbb{G}_m(U')\}/I'_{U'} & \xrightarrow{\tilde{\alpha}} & H(K)\{\mathbb{G}_m(U)\}/I'_U & \xrightarrow{\tilde{\beta}} & H(K)\{\mathbb{G}_m(Y_1 \cap U')\}/I'_{Y_1 \cap U'} \\
\downarrow \cong & & \downarrow & & \downarrow \cong \\
\oplus_{p,q} H^q(U', \mathbb{Z}(p)) & \longrightarrow & \oplus_{p,q} H^q(U, \mathbb{Z}(p)) & \longrightarrow & \oplus_{p,q} H^{q-1}(Y_1 \cap U', \mathbb{Z}(p-1))
\end{array}$$

and since the sequence (1.6) is exact we have to prove that the first row is exact, i.e. if $\iota(x_1) = 0$ then $(\phi_1) \cdot \tilde{\alpha}(x_1)$ is contained in the image of $\tilde{\alpha}$. Then

$$H(K)\{\mathbb{G}_m(U)\}/I'_U \rightarrow H(U)$$

is an isomorphism.

First we prove that the kernel of ι is generated by the elements

$$(3.6) \quad \tilde{R}(f_1, \dots, f_t), \text{ with } f_j = \lambda_j \phi_{i_j}, i_j > 1, \text{ or } f_j = \lambda_j, \\
\text{such that } \sum_j (f_j|_{Y_1 \cap U'}) = 0.$$

We denote by J the ideal generated by elements of the form (3.6) in $H(K)\{\mathbb{G}_m(U')\}$. The image of $I'_{U'} + J$ in $H(K)\{\mathbb{G}_m(Y_1 \cap U')\}$ is equal to $I'_{Y_1 \cap U'}$. In order to see this we note that an element in $I'_{Y_1 \cap U'}$ of the form (3.3) lifts to an element in $I'_{U'}$ (since the restriction $\mathbb{G}_m(U') \rightarrow \mathbb{G}_m(Y_1 \cap U')$ is surjective), and an element in $I'_{Y_1 \cap U'}$ of the form (3.4) with $f_j = \lambda_j \cdot \phi_{i_j}|_{Y_1}$ or $f_j = \lambda_j$ and $\lambda_j \in K^\times$ for every j lifts to an element (3.6) in J .

Thus it is sufficient to prove

$$\ker(\mathbb{G}_m(U') \rightarrow \mathbb{G}_m(Y_1 \cap U')) \subset I'_{U'} + J.$$

It is easy to see that the kernel is generated by elements of the form

- (1) $\lambda \cdot \frac{\phi_i}{\phi_j}$ with i, j such that $Y_1 \cap Y_i = Y_1 \cap Y_j$; $\lambda = \frac{\phi_j|_{Y_1}}{\phi_i|_{Y_1}}$,
- (2) $\lambda \phi_i$ with i such that $Y_i \cap Y_1 = \emptyset$; $\lambda = \frac{1}{\phi_i|_{Y_1}}$.

Since $\left(\frac{\lambda \phi_i}{\phi_j}\right) = \tilde{R}(\lambda \phi_i, -\phi_j) \in J$ and $(\lambda \phi_i) = \tilde{R}(\lambda \phi_i, -1) \in J$ the kernel of ι is J as claimed.

Let $x_1 = \tilde{R}(f_1, \dots, f_t)$ be as in (3.6). We see that $\sum_j f_j = -\mu \cdot \phi_1$ for suitable $\mu \in K$ since every f_j is a polynomial of degree ≤ 1 and the restriction $\sum_j f_j|_{Y_1}$ to Y_1 is vanishing. The case $\mu = 0$ is trivial, thus we may assume $\mu \neq 0$. The following calculation in $H(K)\{\mathbb{G}_m(U)\}/I'_U$ completes

the proof:

$$\begin{aligned}
(\phi_1) \cdot \tilde{\alpha}(x_1) &= (\mu\phi_1) \cdot \tilde{\alpha}(x_1) - (\mu) \cdot \tilde{\alpha}(x_1) \\
&= (\mu\phi_1) \cdot \tilde{R}(f_1, \dots, f_t) + \tilde{R}(\mu\phi_1, f_1, \dots, f_t) - (\mu) \cdot \tilde{R}(f_1, \dots, f_t) \\
&= -(f_1) \cdots (f_t) + [-1/\mu] \cdot \tilde{R}(f_1, \dots, f_t) \\
&\in \text{image}(\tilde{\alpha}).
\end{aligned}$$

□

We know that $H^p(K, \mathbb{Z}(q)) = 0$ if $p > q$, and we have the isomorphism (2.3) with Milnor K -theory. Therefore theorem (3.5) implies the following corollary.

Corollary 3.7. *The morphism of $K_*^M(K)$ algebras*

$$K_*^M(K)\{\mathbb{G}_m(U)\}/J_U \rightarrow \oplus_p H^p(U, \mathbb{Z}(p)); \quad (f) \mapsto [f],$$

is an isomorphism, where the ideal J_U is generated by elements of the form (3.1), (3.2) and (3.3).

3.2. Relation with topological cohomology. In the case $K = \mathbb{C}$ we have a realization functor $DM_{gm, \mathbb{Q}}^{eff, op} \rightarrow D^+(\text{Vec}_{\mathbb{Q}})$ to the derived category of \mathbb{Q} vector spaces ([Hu], 2.1.7), which maps $M_{gm}(X)$ to the singular cochain complex of X^{an} and $\mathbb{Q}(1)$ to $H^1(\mathbb{G}_m^{an}, \mathbb{Q}) \cong \mathbb{Q}$. In particular we get a morphism of rings

$$(3.8) \quad \oplus_p H^p(U, \mathbb{Q}(p)) \rightarrow \oplus_p H^p(U^{an}, \mathbb{Q}).$$

We denote by $S(U)$ the ring $\oplus_p H^p(U, \mathbb{Q}(p))$.

Proposition 3.9. *The homomorphism (3.8) induces an isomorphism of rings*

$$S(U)/(H^1(K, \mathbb{Q}(1)) \cdot S(U)) \xrightarrow{\cong} \oplus_p H^p(U^{an}, \mathbb{Q}).$$

Proof. Using Proposition (1.1) there is an isomorphism $M_{gm}(U) \cong \oplus_{i \in I} \mathbb{Z}(n_i)[n_i]$. This implies that $S(U) \cong \oplus_{i \in I} S(K)[-n_i]$ as $S(K)$ modules and $H^*(U^{an}, \mathbb{Q}) \cong \oplus_{i \in I} \mathbb{Q}[-n_i]$, the map (3.8) being compatible with the decomposition by functoriality. Thus it is sufficient to prove $S(K)/(H^1(K, \mathbb{Q}(1)) \cdot S(K)) \cong \mathbb{Q}$, which follows from the comparison isomorphism (2.3) since $K_*^M(K)$ is generated by $K_1^M(K)$ as algebra. □

Using the realization functor of Ivorra [Iv]:

$$DM_{gm}^{eff}(K)^{op} \rightarrow D^+(K, \mathbb{Z}_l)$$

to the triangulated category of l -adic sheaves defined by Ekedahl [Ek], the statement of Proposition (3.9) for l -adic cohomology can be proved in the same way, with the assumption that K is an algebraically closed field of characteristic $p \neq l$.

3.3. Combinatorial description. When the ground field K is the field of complex numbers \mathbb{C} , we have a combinatorial description of singular cohomology at disposal ([OrSo], Theorem 5.2). In this section we prove that this description holds for the ring

$$A_0 := \oplus_p H^p(U, \mathbb{Z}(p)) / (H^1(K, \mathbb{Z}(1)) \cdot \oplus_p H^p(U, \mathbb{Z}(p)))$$

for every perfect ground field K as well.

Let Q be the cokernel of the inclusion $\mathbb{G}_m(K) \subset \mathbb{G}_m(U)$; by taking divisors $\mathbb{G}_m(U) \xrightarrow{\text{div}} \oplus_{i=1}^r \mathbb{Z} \cdot Y_i$ in \mathbb{A}^N , we get an isomorphism $Q \xrightarrow{\cong} \oplus_{i=1}^r \mathbb{Z} \cdot Y_i$. We denote by $\Lambda_{\mathbb{Z}} Q$ the exterior algebra. Let L be the ideal in $\Lambda_{\mathbb{Z}} Q$ generated by the elements

$$\begin{aligned} & Y_{i_1} \wedge \cdots \wedge Y_{i_s}; \quad \text{if } Y_{i_1} \cap \cdots \cap Y_{i_s} = \emptyset \\ & \sum_{k=1}^s (-1)^k Y_{i_1} \wedge \cdots \wedge \widehat{Y_{i_k}} \wedge \cdots \wedge Y_{i_s}; \quad \text{if } Y_{i_1} \cap \cdots \cap Y_{i_s} \neq \emptyset \\ & \text{and } \text{codim}(Y_{i_1} \cap \cdots \cap Y_{i_s}) < s. \end{aligned}$$

Proposition 3.10. *Let K be a perfect field. The map*

$$\psi : A_0 \rightarrow \Lambda_{\mathbb{Z}} Q / L; \quad [f] \mapsto \text{div}(f)$$

is well-defined and an isomorphism of rings.

Proof. By Corollary (3.7) we have

$$A_0 = K_*^M(K) \{ \mathbb{G}_m(U) \} / (J_U + K^\times \cdot K_*^M(K) \{ \mathbb{G}_m(U) \}).$$

By definition, ψ induces the projection $K_*^M(K) \rightarrow K_0^M$ on $K_*^M(K)$. We have to prove that elements of the form (3.2) and (3.3) map to zero. For elements of the form (3.3) the assertion is trivial and we may replace (3.2) with (3.4) and we may assume $f_j = \lambda_j \phi_{i_j}$ or $f_j = \lambda_j$ for every j and some $\lambda_j \in K^\times$, as in the proof of the theorem (3.5). We need to prove that

$$\alpha := \sum_{k=1}^s (-1)^k \text{div}(f_1) \wedge \cdots \wedge \widehat{\text{div}(f_k)} \wedge \cdots \wedge \text{div}(f_s)$$

is an element of L . Obviously, α is trivial if there are more than two constant functions among the f_1, \dots, f_s ; if f_1 is the only constant function then $Y_{i_2} \cap \cdots \cap Y_{i_s} = \emptyset$ and $\alpha \in L$. In the case of no non-constant function, then either $Y_{i_1} \cap \cdots \cap Y_{i_s} = \emptyset$ or $\text{codim}(Y_{i_1} \cap \cdots \cap Y_{i_s}) < s$. In the first case we have $Y_{i_1} \cap \cdots \cap \widehat{Y_{i_j}} \cap \cdots \cap Y_{i_s} = \emptyset$ for every j and in the second case $\alpha \in L$ by definition of L . This proves that ψ is well-defined.

For the inverse map we have to prove that

$$\Lambda_{\mathbb{Z}}Q/L \rightarrow A_0; \quad Y_i \mapsto [\phi_i],$$

is well-defined. Because A_0 is graded-commutative, the map $\Lambda_{\mathbb{Z}}Q \rightarrow A_0, Y_i \mapsto [\phi_i]$, exists. If $Y_{i_1} \cap \cdots \cap Y_{i_s} = \emptyset$ then $\sum_k \lambda_k \phi_{i_k} = 1$ for some $\lambda_k \in K$ and we have $[\phi_{i_1}] \cdots [\phi_{i_s}] = 0$ in A_0 . If $Y_{i_1} \cap \cdots \cap Y_{i_s} \neq \emptyset$ and $\text{codim}(Y_{i_1} \cap \cdots \cap Y_{i_s}) < s$ then $\sum_k \lambda_k \phi_{i_k} = 0$ for some $\lambda_k \in K$ and $\sum_k (-1)^k [\phi_{i_1}] \cdots [\widehat{\phi_{i_k}}] \cdots [\phi_{i_s}] = 0$ in A_0 . This proves that the inverse map is well-defined. \square

Corollary 3.11. *The rank of the free $H(K)$ module $H(U)$ is equal to the rank of $\Lambda_{\mathbb{Z}}Q/L$.*

Proof. The rank of $H(U)$ is equal to the rank of $\oplus_p H^p(U, \mathbb{Z}(p))$ as $\oplus_p H^p(K, \mathbb{Z}(p)) \cong K_*^M(K)$ module, by Proposition (1.1). Thus, we get

$$\text{rk}_{K_*^M(K)}(\oplus_p H^p(U, \mathbb{Z}(p))) = \text{rk}_{\mathbb{Z}} A_0 = \text{rk}_{\mathbb{Z}} (\Lambda_{\mathbb{Z}}Q/L).$$

\square

The algebra $\Lambda_{\mathbb{Z}}Q/L$ depends only on the combinatorics of the hyperplanes in the complement. For example, assume that

$$(3.12) \quad \overline{Y_1}, \dots, \overline{Y_r}, H \quad \text{is a normal crossing divisor,}$$

where H is the hyperplane at infinity for the compactification $\mathbb{A}^N \subset \mathbb{P}^N$ and $\overline{Y_i}$ is the closure of Y_i in \mathbb{P}^N . Then we have

$$\Lambda_{\mathbb{Z}}Q/L \cong \Lambda_{\mathbb{Z}}Q / \langle Y_{i_1} \wedge \cdots \wedge Y_{i_{N+1}} \mid i_1, \dots, i_{N+1} \rangle,$$

here $\Lambda_{\mathbb{Z}}Q/L$ depends only on (r, N) .

The motivic cohomology ring is a finer invariant. It is easy to see (with assumption (3.12)) that

$$(3.13) \quad \{[\phi_{i_1}] \cdots [\phi_{i_t}] \mid i_1 < \cdots < i_t, t \leq N\} \cup \{1\}$$

is a base of $H(U)$ as module over the motivic cohomology of the ground field. We can define an homomorphism of groups

$$(3.14) \quad \beta : \mathbb{G}_m(U)^{\otimes N+1} \rightarrow K^\times \otimes_{\mathbb{Z}} \Lambda^N Q$$

by setting

$$\beta(f_1 \otimes \cdots \otimes f_{N+1}) = \sum_{i_1 < \cdots < i_N} \alpha_{i_1 < \cdots < i_N} \otimes (Y_{i_1} \wedge \cdots \wedge Y_{i_N})$$

with

$$[f_1] \cdots [f_{N+1}] = \sum_{i_1 < \cdots < i_N} [\alpha_{i_1 < \cdots < i_N}] \cdot [\phi_{i_1}] \cdots [\phi_{i_N}] + \cdots$$

where this expression is the presentation of $[f_1] \cdots [f_{N+1}]$ in the base (3.13). The morphism β is independent of the choices for ϕ_1, \dots, ϕ_r .

For $U = \mathbb{A}^1 - \{p_1, \dots, p_r\}$ it is easy to see that β is the “tame symbol”:

$$\beta(f, g) = \sum_{i=1}^r (-1)^{\nu_{p_i}(f)\nu_{p_i}(g)} \left(\frac{f^{\nu_{p_i}(g)}}{g^{\nu_{p_i}(f)}} \right) (p_i) \otimes p_i.$$

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